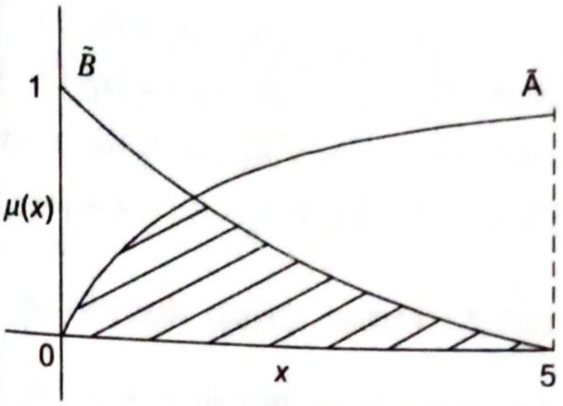
(c) $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

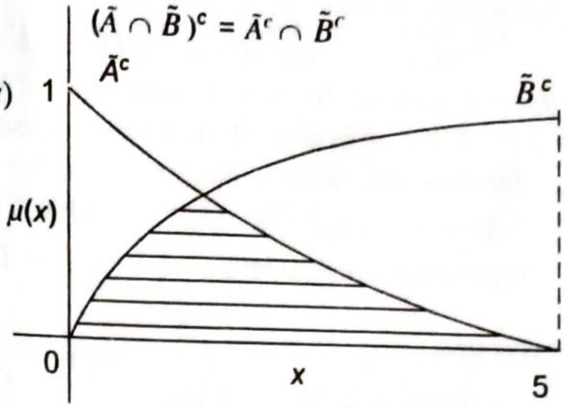
$$= \min\left(\frac{x}{x+1}, 2^{-x}\right)$$



(d) $\mu_{(\tilde{A} \cup \tilde{B})^c}(x) = \mu_{\tilde{A}^c \cap \tilde{B}^c}(x)$ (∵ De Morgan's law)

$$= \min(\mu_{\tilde{A}^c}(x), \mu_{\tilde{B}^c}(x))$$

$$= \min\left(\frac{1}{x+1}, \frac{2^x - 1}{2^x}\right)$$



6.4 CRISP RELATIONS

In this section, we review crisp relations as a prelude to fuzzy relations. The concept of relations between sets is built on the Cartesian product operator of sets.

6.4.1 Cartesian Product

The *Cartesian product* of two sets A and B denoted by $A \times B$ is the set of all ordered pairs such that the first element in the pair belongs to A and the second element belongs to B.

i.e.
$$A \times B = \{(a,b)/a \in A,\ b \in B\}$$

If $A \neq B$ and A and B are non-empty then $A \times B \neq B \times A$.

The Cartesian product could be extended to n number of sets

$$\bigtimes_{i=1}^{n} A_i = \{(a_1, a_2, a_3, \ldots, a_n)/a_i \in A_i \text{ for every } i = 1, 2, \ldots, n\} \tag{6.45}$$

Observe that

$$\left|\bigtimes_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i| \tag{6.46}$$

Example

Given $A_1 = \{a, b\}, A_2 = \{1, 2\}, A_3 = \{\alpha\}$,

$$A_1 \times A_2 = \{(a, 1), (b, 1), (a, 2), (b, 2)\}, \ |A_1 \times A_2| = 4, \text{ and } |A_1| = |A_2| = 2$$

Here, $|A_1 \times A_2| = |A_1| \cdot |A_2|$

Also, $A_1 \times A_2 \times A_3 = \{(a, 1, \alpha), (a, 2, \alpha), (b, 1, \alpha), (b, 2, \alpha)\}$

$$|A_1 \times A_2 \times A_3| = 4 = |A_1| \cdot |A_2| \cdot |A_3|$$

6.4.2 Other Crisp Relations

An *n-ary* relation denoted as $R(X_1, X_2, \ldots, X_n)$ among crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the Cartesian product $\underset{i=1}{\overset{n}{\times}} X_i$ and is indicative of an association or relation among the tuple elements.

For $n = 2$, the relation $R(X_1, X_2)$ is termed as a *binary* relation; for $n = 3$, the relation is termed *ternary*; for $n = 4$, *quarternary*; for $n = 5$, *quinary* and so on.

If the universe of discourse or sets are finite, the *n-ary* relation can be expressed as an *n*-dimensional *relation matrix*. Thus, for a binary relation $R(X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$, the relation matrix R is a two dimensional matrix where X represents the rows, Y represents the columns and $R(i, j) = 1$ if $(x_i, y_j) \in R$ and $R(i, j) = 0$ if $(x_i, y_j) \notin R$.

Example

Given $X = \{1, 2, 3, 4\}$,

$$X \times X = \begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4) \\ (3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4) \end{Bmatrix}$$

Let the relation R be defined as

$$R = \{(x, y)/y = x + 1, \ x, y \in X\}$$

$$R = \{(1, 2)(2, 3)(3, 4)\}$$

The relation matrix R is given by

$$R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

6.4.3 Operations on Relations

Given two relations R and S defined on $X \times Y$ and represented by relation matrices, the following operations are supported by R and S

$R \cup S$

$$R \cup S(x,y) = \max\,(R(x,y), S(x,y)) \tag{6.47}$$

ction: $R \cap S$

$$R \cap S(x,y) = \min\,(R(x,y), S(x,y)) \tag{6.48}$$

lement: $\bar{R}$

$$\bar{R}(x,y) = 1 - R(x,y) \tag{6.49}$$

position of relations: $R \circ S$

R to be a relation on X, Y and S to be a relation on Y, Z then $R \circ S$ is a composition of on on X, Z defined as

$$R \circ S = \{(x,z)/(x,z) \in X \times Z,\ \exists\ y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\} \tag{6.50}$$

mmon form of the composition relation is the *max-min composition.*

-min composition:

en the relation matrices of the relation R and S, the max-min composition is defined as

$$T = R \circ S$$

$$T(x,z) = \max_{y \in Y}(\min(R(x,y), S(y,z))) \tag{6.51}$$

xample

Let R, S be defined on the sets $\{1, 3, 5\} \times \{1, 3, 5\}$

Let

$$R: \{(x, y) \mid y = x + 2\}, \quad S: \{(x, y) \mid x < y\}$$

$$R = \{(1, 3)(3, 5)\}, \qquad S = \{(1, 3)(1, 5)\ (3, 5)\}$$

The relation matrices are

$$R: \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad S: \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

since $R \circ S(1, 1) = \max\{\min(0, 0), \min(1, 0), \min(0, 0)\}$

$= \max(0, 0, 0) = 0.$

$R \circ S(1, 3) = \max\{0, 0, 0\} = 0$

$R \circ S(1, 5) = \max\{0, 1, 0\} = 1.$

Similarly, $R \circ S(3, 1) = 0.$

$R \circ S(3, 3) = R \circ S(3, 5) = R \circ S(5, 1) = R \circ S(5, 3) = R \circ S(5, 5) = 0$

$R \circ S$ from the relation matrix is $\{(1, 5)\}$.

Also,

$$S \circ R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.5 FUZZY RELATIONS

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2, \ldots, X_n$ where the n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership within the relation. The membership values indicate the strength of the relation between the tuples.

Example

Let R be the fuzzy relation between two sets X_1 and X_2 where X_1 is the set of diseases and X_2 is the set of symptoms.

X_1 = {typhoid, viral fever, common cold}

X_2 = {running nose, high temperature, shivering}

The fuzzy relation R may be defined as

	Running nose	*High temperature*	*Shivering*
Typhoid	0.1	0.9	0.8
Viral fever	0.2	0.9	0.7
Common cold	0.9	0.4	0.6

6.5.1 Fuzzy Cartesian Product

Let $\tilde{A}$ be a fuzzy set defined on the universe X and $\tilde{B}$ be a fuzzy set defined on the universe Y. the Cartesian product between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$ and resulting in a fuzzy relation $\tilde{R}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y$$

where $\tilde{R}$ has its membership function given by (6.52)

$$\mu_{\tilde{R}}(x, y) = \mu_{\tilde{A}\times\tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

(6.53)

Example

Let $\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ and $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the universes of discourse $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Then the fuzzy relation $\tilde{R}$ resulting out of the fuzzy Cartesian product $\tilde{A}\times\tilde{B}$ is given by

$$\tilde{R} = \tilde{A}\times\tilde{B} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

since,

$$\tilde{R}(x_1, y_1) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1)) = \min(0.2, 0.5) = 0.2$$
$$\tilde{R}(x_1, y_2) = \min(0.2, 0.6) = 0.2$$
$$\tilde{R}(x_2, y_1) = \min(0.7, 0.5) = 0.5$$
$$\tilde{R}(x_2, y_2) = \min(0.7, 0.6) = 0.6$$
$$\tilde{R}(x_3, y_1) = \min(0.4, 0.5) = 0.4$$
$$\tilde{R}(x_3, y_2) = \min(0.4, 0.6) = 0.4$$

6.5.2 Operations on Fuzzy Relations

Let $\tilde{R}$ and $\tilde{S}$ be fuzzy relations on $X \times Y$.

Union (6.54)

$$\mu_{\tilde{R}\cup\tilde{S}}(x, y) = \max(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$$

Intersection (6.55)

$$\mu_{\tilde{R}\cap\tilde{S}}(x, y) = \min(\mu_{\tilde{R}}(x, y), \mu_{\tilde{S}}(x, y))$$

Complement (6.56)

$$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y)$$

Composition of relations

The definition is similar to that of crisp relation. Suppose $\tilde{R}$ is a fuzzy relation defined on $X \times Y$, and $\tilde{S}$ is a fuzzy relation defined on $Y \times Z$, then $\tilde{R} \circ \tilde{S}$ is a fuzzy relation on $X \times Z$. The fuzzy max-min composition is defined as

$$\mu_{\tilde{R}\circ\tilde{S}}(x,z) = \max_{y \in Y}(\min(\mu_{\tilde{R}}(x,y),\ \mu_{\tilde{S}}(y,z)))$$

Example

$$X = \{x_1, x_2, x_3\}\ Y = \{y_1, y_2\}\ Z = \{z_1, z_2, z_3\}$$

Let $\tilde{R}$ be a fuzzy relation

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

Let $\tilde{S}$ be a fuzzy relation

$$\begin{array}{c} \\ y_1 \\ y_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

Then $R \circ S$, by max-min composition yields,

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_1) &= \max(\min(0.5, 0.6), \min(0.1, 0.5)) \\ &= \max(0.5, 0.1) \\ &= 0.5. \end{aligned}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_2) &= \max(\min(0.5, 0.4), \min(0.1, 0.8)) \\ &= \max(0.4, 0.1) \\ &= 0.4 \end{aligned}$$

Similarly,

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1,z_3) = \max(0.5, 0.1) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_1) = \max(0.2, 0.5) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_2) = \max(0.2, 0.8) = 0.8$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_3) = \max(0.2, 0.9) = 0.9$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_1) = \max(0.6, 0.5) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_2) = \max(0.4, 0.6) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_3) = \max(0.7, 0.6) = 0.7$$

Example 6.7

Consider a set $P = \{P_1, P_2, P_3, P_4\}$ of four varieties of paddy plants, set $D = \{D_1, D_2, D_3, D_4\}$ of the various diseases affecting the plants and $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let $\tilde{R}$ be a relation on $P \times D$ and $\tilde{S}$ be a relation on $D \times S$

For,

$$\tilde{R} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.1 & 0.2 \end{bmatrix} \end{array} \qquad \tilde{S} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1 & 1 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.9 \\ 0.9 & 1 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of the plants with the different symptoms of the diseases using max-min composition.

Solution

To obtain the association of the plants with the symptoms, $R \circ S$ which is a relation on the sets P and S is to be computed.

Using max-min composition,

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

SUMMARY

- Fuzzy set theory is an effective tool to tackle the problem of uncertainty.
- In crisp logic, an event can take on only two values, either a 1 or 0 depending on whether its occurrence is true or false respectively. However, in fuzzy logic, the event may take a range of values between 0 and 1.
- Crisp sets are fundamental to the study of fuzzy sets. The basic concepts include universal set, membership, cardinality of a set, family of sets, Venn diagrams, null set singleton set, power set, subset, and super set. The basic operations on crisp sets ar union, intersection, complement, and difference. A set of properties are satisfied by cris sets. Also, the concept of partition and covering result in the two important rules, namel rule of addition and principle of inclusion and exclusion.
- Fuzzy sets support a flexible sense of membership and is defined to be the pair $(x, \mu_{\tilde{A}}($ where $\mu_{\tilde{A}}(x)$ could be discrete or could be described by a continuous function. T membership functions could be triangular, trapezoidal, curved or its variations.

Chapter 7

Fuzzy Systems

Logic is the science of reasoning. Symbolic or *mathematical* logic has turned out to be a powerful computational paradigm. Not only does symbolic logic help in the description of events in the real world but has also turned out to be an effective tool for inferring or deducing information from a given set of facts.

Just as mathematical sets have been classified into crisp sets and fuzzy sets (Refer Chapter 6), logic can also be broadly viewed as *crisp logic* and *fuzzy logic*. Just as crisp sets survive on a 2-state membership (0/1) and fuzzy sets on a multistate membership [0–1], crisp logic is built on a 2-state truth value (True/False) and fuzzy logic on a multistate truth value (True/False/very True/partly False and so on.)

We now briefly discuss crisp logic as a prelude to fuzzy logic.

7.1 CRISP LOGIC

Consider the statements "Water boils at 90°C" and "Sky is blue". An agreement or disagreement with these statements is indicated by a "True" or "False" value accorded to the statements. While the first statement takes on a value *false*, the second takes on a value *true*.

Thus, a statement which is either 'True' or 'False' but not both is called a *proposition*. A proposition is indicated by upper case letters such as P, Q, R and so on.

Example: P: Water boils at 90°C.

Q: Sky is blue.

are propositions.

A simple proposition is also known as an *atom*. Propositions alone are insufficient to represent phenomena in the real world. In order to represent complex information, one has to build a sequence of propositions linked using *connectives* or *operators*. Propositional logic recognizes five major operators as shown in Table 7.1.

Table 7.1 Propositional logic connectives

Symbol	Connective	Usage	Description
$\wedge$	and	$P \wedge Q$	P and Q are true.
$\vee$	or	$P \vee Q$	Either P or Q is true.
$\neg$ or $\sim$	not	$\sim P$ or $\neg P$	P is not true.
$\Rightarrow$	implication	$P \Rightarrow Q$	P implies Q is true.
$=$	equality	$P = Q$	P and Q are equal (in truth values) is true.

Observe that ∧, ∨, ⇒, and = are 'binary' operators requiring two propositions wh 'unary' operator requiring a single proposition. ∧ and ∨ operations are referred to as co and *disjunction* respectively. In the case of ⇒ operator, the proposition occurring before symbol is called as the *antecedent* and the one occurring after is called as the conseq

The semantics or meaning of the logical connectives are explained using a truth truth table comprises rows known as *interpretations*, each of which evaluates the logical for the given set of truth values. Table 7.2 illustrates the truth table for the five connec

Table 7.2 Truth table for the connectives ∧, ∨, ~, ⇒ , =

P	*Q*	*P* ∧ *Q*	*P* ∨ *Q*	~*P*	*P* ⇒ *Q*	*P* = *Q*
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	F	F	F	T	T	T
F	T	F	T	T	T	F

T : True, F : False

A logical formula comprising *n* propositions will have 2^n interpretations in its truth t formula which has all its interpretations recording true is known as a *tautology* and the one records false for all its interpretations is known as *contradiction*.

Example 7.1

Obtain a truth table for the formula (*P* ∨ *Q*) ⇒ (~*P*). Is it a tautology?

Solution

The truth table for the given formula is

P	*Q*	*P* ∨ *Q*	~*P*	*P* ∨ *Q* ⇒ ~*P*
T	F	T	F	F
F	T	T	T	T
T	T	T	F	F
F	F	F	T	T

No, it is not a tautology since all interpretations do not record 'True' in its last column.

Example 7.2

Is ((*P* ⇒ *Q*) ∧ (*Q* ⇒ *P*) = (*P* = *Q*) a tautology?

Solution

P	*Q*	*P* ⇒ *Q*	*Q* ⇒ *P*	*A*: (*P* ⇒ *Q*) ∧ (*Q* ⇒ *P*)	*B*: *P* = *Q*	*A* = *B*
T	F	F	T	F	F	T
F	T	T	F	F	F	T
T	T	T	T	T	T	T
F	F	T	T	T	T	T

Yes, the given formula is a tautology.

Example 7.3

Show that $(P \Rightarrow Q) = (\sim P \vee Q)$

Solution

The truth table for the given formula is

P	Q	$A: P \Rightarrow Q$	$\sim P$	$B: \sim P \vee Q$	$A = B$
T	T	T	F	T	T
T	F	F	F	F	T
F	F	T	T	T	T
T	T	T	T	T	T

Since the last column yields 'True' for all interpretations, it is a tautology.

The logical formula presented in Example 7.3 is of practical importance since $(P \Rightarrow Q)$ is shown to be equivalent to $(\sim P \vee Q)$, a formula devoid of '$\Rightarrow$' connective. This equivalence can therefore be utilised to eliminate '$\Rightarrow$' in logical formulae.

It is useful to view the '$\Rightarrow$' operator from a set oriented perspective. If X is the universe of discourse and A, B are sets defined in X, then propositions P and Q could be defined based on an element $x \in X$ belonging to A or B. That is,

$$P: x \in A$$

$$Q: x \in B \tag{7.1}$$

Here, P, Q are true if $x \in A$ and $x \in B$ respectively, and $\sim P$, $\sim Q$ are true if $x \notin A$ and $x \notin B$ respectively. In such a background, $P \Rightarrow Q$ which is equivalent to $(\sim P \vee Q)$ could be interpreted as

$$(P \Rightarrow Q) : x \notin A \quad \text{or} \quad x \in B \tag{7.2}$$

However, if the '$\Rightarrow$' connective deals with two different universes of discourse, that is, $A \subset X$ and $B \subset Y$ where X and Y are two universes of discourse then the '$\Rightarrow$' connective is represented by the relation R such that

$$R = (A \times B) \cup (\bar{A} \times Y) \tag{7.3}$$

In such a case, $P \Rightarrow Q$ is linguistically referred to as IF A THEN B. The compound proposition $(P \Rightarrow Q) \vee (\sim P \Rightarrow S)$ linguistically referred to as IF A THEN B ELSE C is equivalent to

$$\text{IF } A \text{ THEN } B \ (P \Rightarrow Q)$$

$$\text{IF } \sim A \text{ THEN } C \ (\sim P \Rightarrow S) \tag{7.4}$$

where P, Q, and S are defined by sets A, B, C, $A \subset X$, and B, $C \subset Y$.

7.1.1 Laws of Propositional Logic

Crisp sets as discussed in Section 6.2.2. exhibit properties which help in their simplificatio

Similarly, propositional logic also supports the following laws which can be effectively u… their simplification. Given P, Q, R to be the propositions,

(i) *Commutativity*

$$(P \vee Q) = (Q \vee P)$$
$$(P \wedge Q) = (Q \wedge P)$$

(ii) *Associativity*

$$(P \vee Q) \vee R = P \vee (Q \vee R)$$
$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$$

(iii) *Distributivity*

$$(P \vee Q) \wedge R = (P \wedge R) \vee (Q \wedge R)$$
$$(P \wedge Q) \vee R = (P \vee R) \wedge (Q \vee R)$$

(iv) *Identity*

$$P \vee \text{false} = P$$
$$P \wedge \text{True} = P$$
$$P \wedge \text{False} = \text{False}$$
$$P \vee \text{True} = \text{True}$$

(v) *Negation*

$$P \wedge \sim P = \text{False}$$
$$P \vee \sim P = \text{True}$$

(vi) *Idempotence*

$$P \vee P = P$$
$$P \wedge P = P$$

(vii) *Absorption*

$$P \wedge (P \vee Q) = P$$
$$P \vee (P \wedge Q) = P$$

(viii) *De Morgan's laws*

$$\sim(P \vee Q) = (\sim P \wedge \sim Q)$$
$$\sim(P \wedge Q) = (\sim P \vee \sim Q)$$

(ix) *Involution*

$$\sim(\sim P) = P$$

Each of these laws can be tested to be a tautology using truth tables.

Example 7.4

Verify De Morgan's laws.

(a) $\sim(P \vee Q) = (\sim P \wedge \sim Q)$

(b) $\sim(P \wedge Q) = (\sim P \vee \sim Q)$

Solution

(a)

P	Q	$P \vee Q$	$A: \sim(P \vee Q)$	$\sim P$	$\sim Q$	$B: \sim P \wedge \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Therefore, $\sim(P \vee Q) = (\sim P \wedge \sim Q)$

(b)

P	Q	$P \wedge Q$	$A: \sim(P \wedge Q)$	$\sim P$	$\sim Q$	$B: \sim P \vee \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	T	T	T
F	F	F	T	T	F	T	T

Therefore $\sim(P \wedge Q) = (\sim P \vee \sim Q)$

Example 7.5

Simplify $(\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

Solution

Consider $(\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

$= (\sim\sim(P \wedge Q) \vee R) \wedge P \wedge Q$

(by eliminating '$\Rightarrow$' using $(P \Rightarrow Q) = (\sim P \vee Q)$)

$= ((P \wedge Q) \vee R) \wedge P \wedge Q$ (by the law of involution)

$= (P \wedge Q)$ (by the law of absorption)

7.1.2 Inference in Propositional Logic

Inference is a technique by which, given a set of *facts* or *postulates* or *axioms* or *premises* $F_1, F_2, \ldots, F_n$, a *goal G* is to be derived. For example, from the facts "Where there is smoke there is fire", and "There is smoke in the hill", the statement "Then the hill is on fire" can be easily deduced.

In propositional logic, three rules are widely used for inferring facts, namely

(i) *Modus Ponens*

(ii) *Modus Tollens*, and

(iii) *Chain rule*

Modus ponens (mod pons)

Given $P \Rightarrow Q$ and P to be true, Q is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ P \end{array}}{Q}$$

Here, the formulae above the line are the *premises* and the one below is the goal which can be inferred from the premises.

Modus tollens

Given $P \Rightarrow Q$ and $\sim Q$ to be true, $\sim P$ is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ \sim Q \end{array}}{\sim P}$$

Chain rule

Given $P \Rightarrow Q$ and $Q \Rightarrow R$ to be true, $P \Rightarrow R$ is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \end{array}}{P \Rightarrow R}$$

Note that the chain rule is a representation of the *transitivity* relation with respect to the '⇒' connective.

Example 7.6

Given

(i) $C \vee D$

(ii) $\sim H \Rightarrow (A \wedge \sim B)$

(iii) $(C \vee D) \Rightarrow \sim H$

(iv) $(A \wedge \sim B) \Rightarrow (R \vee S)$

Can $(R \vee S)$ be inferred from the above?

Solution

From (i) and (iii) using the rule of Modus Ponens, ~*H* can be inferred.

$$\begin{array}{ll} \text{(i)} & C \vee D \\ \text{(iii)} & (C \vee D) \Rightarrow \sim H \\ \hline & \sim H \quad \text{(v)} \end{array}$$

From (ii) and (iv) using the chain rule, ~*H* ⇒ (*R* ∨ *S*) can be inferred.

$$\begin{array}{ll} \text{(ii)} & \sim H \Rightarrow (A \wedge \sim B) \\ \text{(iv)} & (A \wedge \sim B) \Rightarrow (R \vee S) \\ \hline & \sim H \Rightarrow (R \vee S) \quad \text{(vi)} \end{array}$$

From (v) and (vi) using the rule of Modus Ponens (*R* ∨ *S*) can be inferred.

$$\begin{array}{ll} \text{(vi)} & \sim H \Rightarrow (R \vee S) \\ \text{(v)} & \sim H \\ \hline & R \vee S \end{array}$$

Hence, the result.

7.2 PREDICATE LOGIC

In propositional logic, events are symbolised as propositions which acquire either 'True/False' values. However, there are situations in the real world where propositional logic falls short of its expectation. For example, consider the following statements:

P : All men are mortal.

Q : Socrates is a man.

From the given statements it is possible to infer that Socrates is mortal. However, from the propositions *P*, *Q* which symbolise these statements nothing can be made out. The reason being, propositional logic lacks the ability to symbolise *quantification*. Thus, in this example, the quantifier "All" which represents the entire class of men encompasses Socrates as well, who is declared to be a man, in proposition *Q*. Therefore, by virtue of the first proposition *P*, Socrates who is a man also becomes a mortal, giving rise to the deduction Socrates is mortal. However, the deduction is not directly perceivable owing to the shortcomings in propositional logic. Therefore, propositional logic needs to be augmented with more tools to enhance its logical abilities.

Predicate logic comprises the following apart from the connectives and propositions recognized by propositional logic.

(i) Constants

(ii) Variables

Example

plus (2, 3) (2 plus 3 which is 5)

mother (Krishna) (Krishna's mother)

Observe that plus () and mother () indirectly describe "5" and "Krishna's mother" respectively.

Example 7.7

Write predicate logic statements for

(i) Ram likes all kinds of food.

(ii) Sita likes anything which Ram likes.

(iii) Raj likes those which Sita and Ram both like.

(iv) Ali likes some of which Ram likes.

Solution

Let food (x) : x is food.

likes (x, y) : x likes y

Then the above statements are translated as

(i) $\forall x$ food $(x) \Rightarrow$ likes (Ram, x))

(ii) $\forall x$ (likes (Ram, x) $\Rightarrow$ likes (Sita, x))

(iii) $\forall x$ (likes (Sita, x) $\wedge$ likes (Ram, x)) $\Rightarrow$ likes (Raj, x))

(iv) $\exists x$ (likes (Ram, x) $\wedge$ likes (Ali, x))

The application of the rule of universal quantifier and rule of existential quantifier can be observed in the translations given above.

7.2.1 Interpretations of Predicate Logic Formula

For a formula in propositional logic, depending on the truth values acquired by the propositions, the truth table interprets the formula. But in the case of predicate logic, depending on the truth values acquired by the predicates, the nature of the quantifiers, and the values taken by the constants and functions over a domain D, the formula is interpreted.

Example

Interpret the formulae

(i) $\forall x\ p(x)$

(ii) $\exists x\ p(x)$

where the domain $D = \{1, 2\}$ and

$p(1)$	$p(2)$
True	False

Solution

(i) $\forall x\ p(x)$ is true only if $p(x)$ is true for all values of x in the domain D, other
Here, for $x = 1$ and $x = 2$, the two possible values for x chosen from D, nam
and $p(2)$ = false respectively, yields (i) to be false since $p(x)$ is not true for
$\forall x\ p(x)$ is false.

(ii) $\exists x\ p(x)$ is true only if there is atleast one value of x for which $p(x)$ is true
Here, for $x = 1$, $p(x)$ is true resulting in (ii) to be true. Hence, $\exists x\ p(x)$ is tr

Example 7.8

Interpret $\forall x\ \exists y\ P(x, y)$ for $D = \{1, 2\}$ and

$P(1, 1)$	$P(1, 2)$	$P(2, 1)$	$P(2, 2)$
True	False	False	True

Solution

For $x = 1$, there exists a y, $(y = 1)$ for which $P(x, y)$, i.e. $(P(1,1))$ is true.

For $x = 2$, there exists a y, $(y = 2)$ for which $P(x, y)$ $(P(2, 2))$ is true.

Thus, for all values of x there exists a y for which $P(x, y)$ is true.

Hence, $\forall x\ \exists y\ P(x, y)$ is true.

7.2.2 Inference in Predicate Logic

The rules of inference such as Modus Ponens, Modus Tollens and Chain rule, and th
propositional logic are applicable for inferring predicate logic but not before the quanti
been appropriately eliminated (refer Chang & Lee, 1973).

Example

Given (i) All men are mortal.
(ii) Confucius is a man.
Prove: Confucius is mortal.

Translating the above into predicate logic statements

(i) $\forall x\ (\text{man}\ (x) \Rightarrow \text{mortal}\ (x))$

(ii) man (Confucius)

(iii) mortal (Confucius)

Since (i) is a tautology qualified by the universal quantifier for x = Confucius, the stater
true, i.e.

$$\text{man (Confucius)} \Rightarrow \text{mortal (Confucius)}$$
$$\Rightarrow \sim\text{man (Confucius)} \vee \text{mortal (Confucius)}$$

But from (ii), man (Confucius) is true.
Hence (iv) simplifies to

$$\text{False} \vee \text{mortal (Confucius)}$$
$$= \text{mortal (Confucius)}$$

Hence, Confucius is mortal has been proved.